

Exponential sums on \mathbf{A}^n , III

Alan Adolphson and Steven Sperber

ABSTRACT. We give two applications of our earlier work[4]. We compute the p -adic cohomology of certain exponential sums on \mathbf{A}^n involving a polynomial whose homogeneous component of highest degree defines a projective hypersurface with at worst weighted homogeneous isolated singularities. This study was motivated by recent work of García[9]. We also compute the p -adic cohomology of certain exponential sums on \mathbf{A}^n whose degree is divisible by the characteristic.

1. Introduction

Let p be a prime number, $q = p^a$, and \mathbf{F}_q the finite field of q elements. Associated to a polynomial $f \in \mathbf{F}_q[x_1, \dots, x_n]$ and a nontrivial additive character $\Psi : \mathbf{F}_q \rightarrow \mathbf{C}^\times$ are exponential sums

$$(1.1) \quad S(\mathbf{A}^n(\mathbf{F}_{q^i}), f) = \sum_{x_1, \dots, x_n \in \mathbf{F}_{q^i}} \Psi(\text{Trace}_{\mathbf{F}_{q^i}/\mathbf{F}_q} f(x_1, \dots, x_n))$$

and an L -function

$$(1.2) \quad L(\mathbf{A}^n, f; t) = \exp\left(\sum_{i=1}^{\infty} S(\mathbf{A}^n(\mathbf{F}_{q^i}), f) \frac{t^i}{i}\right).$$

Dwork has associated to f a complex $(\Omega_{C(b)}, D)$ (of length n), depending on a choice of rational parameter b satisfying $0 < b < p/(p-1)$ (see [3] for details). Each cohomology group $H^i(\Omega_{C(b)}, D)$ is a vector space over a field $\tilde{\Omega}_0$ (a finite extension of \mathbf{Q}_p) and has a Frobenius operator F satisfying

$$L(\mathbf{A}^n, f; t) = \prod_{i=0}^n \det(I - tF \mid H^i(\Omega_{C(b)}, D))^{(-1)^{i+1}}.$$

We write $\mathbf{F}_q[x]$ for $\mathbf{F}_q[x_1, \dots, x_n]$ and consider the complex $(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k, \phi_f)$, where $\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k$ denotes the module of differential k -forms of $\mathbf{F}_q[x_1, \dots, x_n]$ over \mathbf{F}_q and $\phi_f : \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k \rightarrow \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k+1}$ is defined by

$$\phi_f(\omega) = df \wedge \omega,$$

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where $d : \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k \rightarrow \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k+1}$ is the exterior derivative. Every $\omega \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k$ can be uniquely written in the form

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} \omega(i_1, \dots, i_k) dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

with $\omega(i_1, \dots, i_k) \in \mathbf{F}_q[x]$. If each coefficient $\omega(i_1, \dots, i_k)$ is a homogeneous form of degree l , we call ω *homogeneous* and define

$$\deg \omega = l + (n - k)(\delta - 1),$$

where $\delta = \deg f$. The point of this definition is that we can define an increasing filtration F_\cdot on $\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k$ by setting

$$F_l \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k = \text{the } \mathbf{F}_q\text{-span of homogeneous } k\text{-forms } \omega \text{ with } \deg \omega \leq l,$$

and $(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k, \phi_f)$ then becomes a filtered complex. Consider the associated spectral sequence

$$(1.3) \quad E_1^{r,s} = H^{r+s}(F_r/F_{r-1}(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k, \phi_f)) \Rightarrow H^{r+s}(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k, \phi_f).$$

As an immediate consequence of [4, Theorem 1.13], we have the following.

THEOREM 1.4. *Suppose there exists a positive integer e satisfying*

$$(1.5) \quad \left(1 + \frac{p}{(p-1)^2}\right)(e-1) < \delta$$

such that $E_e^{r,s} = 0$ for all r, s with $r + s \neq n$. Then for

$$(1.6) \quad \frac{\delta}{(p-1)(\delta-e+1)} < b < \frac{p\delta}{(p-1)\delta+e-1}$$

we have

$$(1.7) \quad H^i(\Omega_{C(b)}, D) = 0 \quad \text{for } i \neq n$$

and

$$(1.8) \quad \dim_{\tilde{\Omega}_0} H^n(\Omega_{C(b)}, D) = M_f,$$

where M_f is the sum of the Milnor numbers of the critical points of the mapping $f : \mathbf{A}^n \rightarrow \mathbf{A}^1$. In particular, $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree M_f .

Remark. Inequality (1.5) is equivalent to the assertion that the right-most term in (1.6) is greater than the left-most term in (1.6), i. e., (1.5) is equivalent to the existence of rational b satisfying (1.6). It is explained in [4, section 1] that the vanishing of $E_e^{r,s}$ for all r, s with $r + s \neq n$ implies that $f : \mathbf{A}^n \rightarrow \mathbf{A}^1$ has isolated critical points, hence the sum of the Milnor numbers is finite.

To apply Theorem 1.4, one must find conditions on the polynomial f that guarantee that, for some $e \geq 1$, $E_e^{r,s} = 0$ for all r, s with $r + s \neq n$. When $e = 1$, this is equivalent to the condition that the partial derivatives of the homogeneous component of degree δ of f form a regular sequence in $\mathbf{F}_q[x]$. When $e > 1$, the problem is much harder. We gave one example of such a condition in [4, section 5]. The purpose of this article is to give two more examples of such conditions.

Write

$$(1.9) \quad f = f^{(\delta)} + f^{(\delta')} + f^{(\delta'-1)} + \dots + f^{(0)},$$

where $f^{(i)}$ is homogeneous of degree i and $1 \leq \delta' \leq \delta - 1$, i. e., $f^{(\delta')}$ is the homogeneous component of second-highest degree of f . We prove the following result, which was stated in [4]. (The terms “weighted homogeneous” isolated singularity and “total degree” of a weighted homogeneous isolated singularity will be defined in the next section.)

THEOREM 1.10. *Suppose that the hypersurface $f^{(\delta)} = 0$ in \mathbf{P}^{n-1} has at worst weighted homogeneous isolated singularities, of total degrees $\delta_1, \dots, \delta_s$, and that none of these singularities lies on the hypersurface $f^{(\delta')} = 0$ in \mathbf{P}^{n-1} . Suppose also that $(p, \delta\delta'\delta_1 \cdots \delta_s) = 1$. Then $E_{\delta-\delta'+1}^{r,s} = 0$ for all r, s with $r + s \neq n$.*

The hypothesis of Theorem 1.10, with $\delta' = \delta - 1$, was first considered by García[9]. He showed in that case that the l -adic cohomology groups of the exponential sum (1.1) vanish except in degree n and that the reciprocal roots of $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ are pure of weight n . In particular, he obtains the estimate

$$|S(\mathbf{A}^n(\mathbf{F}_{q^i}), f)| \leq M_f q^{ni/2}.$$

By our approach, we have not been able to obtain archimedean estimates for the reciprocal roots of $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$. Thus that question is still open for $\delta' < \delta - 1$, although we conjecture the roots are again pure of weight n when (1.5) holds for $e = \delta - \delta' + 1$.

For our second example, we consider the case where the degree of f is divisible by p . This uses ideas similar to those in the proof of Theorem 1.10, but the computations are simpler. As we noted in [3], if $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n$ form a regular sequence in $\mathbf{F}_q[x]$, then $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree $(\delta - 1)^n$ all of whose reciprocal roots have absolute value $q^{n/2}$, even if $p|\delta$. We consider the case where these partial derivatives do not form a regular sequence.

THEOREM 1.11. *Suppose $p|\delta$ and the set of common zeros of $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n$ in \mathbf{P}^{n-1} is finite and nonempty. Suppose also that $(p, \delta') = 1$ and that the hypersurface $f^{(\delta')} = 0$ in \mathbf{P}^{n-1} contains no common zero of $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n$. Then $E_{\delta-\delta'+1}^{r,s} = 0$ for all r, s with $r + s \neq n$.*

Remark. For example, if $p|\delta$ and $f^{(\delta)} = 0$ defines a smooth hypersurface in \mathbf{P}^{n-1} , then the set of common zeroes of $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n$ in \mathbf{P}^{n-1} is finite. (If the set of common zeros had dimension ≥ 1 , it would have nonempty intersection with the hypersurface $f^{(\delta)} = 0$, and any such point of intersection would be a singular point of this hypersurface.) We conjecture that under the hypothesis of Theorem 1.11, the reciprocal roots of $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ are pure of weight n when (1.5) holds for $e = \delta - \delta' + 1$.

García also gave a formula for the degree of $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ in terms of the Milnor numbers of the singularities of $f^{(\delta)} = 0$. We derive an analogous formula in section 6 under the hypothesis of either Theorem 1.10 or 1.11.

For certain of the constructions made in the proofs of Theorems 1.10 and 1.11, it may be necessary to extend scalars from \mathbf{F}_q to a larger finite field. Since such extensions of scalars do not affect the computation of cohomology, we make no further comment on them.

2. Hypersurface singularities

For this general discussion of singularities, we work over an arbitrary algebraically closed field K . Let $f \in K[x_1, \dots, x_n]$, put $\mathbf{0} = (0, \dots, 0)$, and assume

$f(\mathbf{0}) = 0$. We say that the hypersurface $f = 0$ has an *isolated singularity at $\mathbf{0}$* if $\mathbf{0}$ is an isolated critical point of the map $f : K^n \rightarrow K$, i. e., there exists a Zariski open neighborhood U of $\mathbf{0}$ in K^n such that the only common zero of $\partial f/\partial x_1, \dots, \partial f/\partial x_n$ on U is $\mathbf{0}$. Let $\mathbf{m} = (x_1, \dots, x_n)$, the maximal ideal of $K[x_1, \dots, x_n]$ corresponding to $\mathbf{0}$ and let $K[x_1, \dots, x_n]_{\mathbf{m}}$ be the localization of $K[x_1, \dots, x_n]$ at \mathbf{m} . When $\mathbf{0}$ is an isolated singularity, then

$$(2.1) \quad \text{Krull dim } K[x_1, \dots, x_n]_{\mathbf{m}}/(\partial f/\partial x_1, \dots, \partial f/\partial x_n) = 0,$$

hence

$$\dim_K K[x_1, \dots, x_n]_{\mathbf{m}}/(\partial f/\partial x_1, \dots, \partial f/\partial x_n) < \infty.$$

This dimension is called the *Milnor number* μ of the isolated singularity. We note that (2.1) implies that $\partial f/\partial x_1, \dots, \partial f/\partial x_n$ generate an \mathbf{m} -primary ideal in $K[x_1, \dots, x_n]_{\mathbf{m}}$, hence form a regular sequence in that ring.

When $\text{char } K = 0$, this definition of isolated singularity is equivalent to the condition that the hypersurface $f = 0$ be nonsingular in a punctured Zariski neighborhood of $\mathbf{0}$ on that hypersurface, i. e., that

$$(2.2) \quad \text{Krull dim } K[x_1, \dots, x_n]_{\mathbf{m}}/(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n) = 0.$$

For by the Theorem of Sard-Bertini, the hypersurface $f = c$ is nonsingular except for finitely many $c \in K$, hence by omitting finitely many hypersurfaces one obtains a Zariski neighborhood of $\mathbf{0}$ in K^n in which $\mathbf{0}$ is the only critical point of the map f . If $\text{char } K = p > 0$, the Theorem of Sard-Bertini fails and condition (2.2) does not imply that $\mathbf{0}$ is an isolated singularity. For example, take $f = x_1^p + x_2^a$ with $(p, a) = 1$. Then (2.2) holds but f has infinitely many critical points in any Zariski neighborhood of $\mathbf{0}$ in K^2 and

$$\text{Krull dim } K[x_1, x_2]_{\mathbf{m}}/(\partial f/\partial x_1, \partial f/\partial x_2) = 1.$$

Recall that $g \in K[x_1, \dots, x_n]$ is called *weighted homogeneous of total degree δ* if there exist positive integers $\alpha_1, \dots, \alpha_n$ with greatest common divisor 1 such that

$$g(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n) = \lambda^{\delta} g(x_1, \dots, x_n).$$

When this holds, we also have the Euler-type relation

$$(2.3) \quad \delta g = \sum_{i=1}^n \alpha_i x_i \frac{\partial g}{\partial x_i}.$$

Note that δ and the α_i may not be uniquely determined. For example, if $g(x_1, x_2) = x_1 x_2$, then $g(\lambda x_1, \lambda x_2) = \lambda^2 g(x_1, x_2)$ and $g(\lambda x_1, \lambda^2 x_2) = \lambda^3 g(x_1, x_2)$.

We say that the isolated singularity $\mathbf{0}$ of the hypersurface $f = 0$ is *weighted homogeneous* if there exists a weighted homogeneous polynomial g such that

$$K[[x_1, \dots, x_n]]/(f) \simeq K[[x_1, \dots, x_n]]/(g).$$

A total degree δ of g is called a *total degree* of the isolated singularity $\mathbf{0}$. In this situation, there exists a regular system of parameters $x'_1, \dots, x'_n \in K[[x_1, \dots, x_n]]$ (i. e., n elements of $K[[x_1, \dots, x_n]]$ that generate its maximal ideal) such that

$$(2.4) \quad f(x_1, \dots, x_n) = g(x'_1, \dots, x'_n).$$

This follows from [10, Lemma 1.7], whose proof is valid over an arbitrary field.

Note that (2.1) implies that when $\mathbf{0}$ is an isolated singularity of $f = 0$, there exists a positive integer m such that

$$f^m \in (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$$

in the local ring $K[x_1, \dots, x_n]_{\mathbf{m}}$.

LEMMA 2.5. *Suppose $\mathbf{0}$ is a weighted homogeneous isolated singularity of the hypersurface $f = 0$. If $\text{char } K = p > 0$, assume also that $(p, \delta) = 1$, where δ is a total degree of $\mathbf{0}$. Then*

$$f \in (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$$

in the local ring $K[x_1, \dots, x_n]_{\mathbf{m}}$. Furthermore, in every representation

$$f = \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}$$

in $K[x_1, \dots, x_n]_{\mathbf{m}}$, h_1, \dots, h_n must lie in the maximal ideal of $K[x_1, \dots, x_n]_{\mathbf{m}}$.

Proof. From (2.3) it follows that

$$g \in (\partial g / \partial x_1, \dots, \partial g / \partial x_n)$$

in $K[x_1, \dots, x_n]$. Equation (2.4) then implies that

$$f \in (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$$

in $K[[x_1, \dots, x_n]]$. But the natural inclusion

$$K[x_1, \dots, x_n]_{\mathbf{m}} \hookrightarrow K[[x_1, \dots, x_n]]$$

induces an isomorphism

$$K[x_1, \dots, x_n]_{\mathbf{m}} / (\partial f / \partial x_1, \dots, \partial f / \partial x_n) \simeq K[[x_1, \dots, x_n]] / (\partial f / \partial x_1, \dots, \partial f / \partial x_n).$$

This implies the first assertion of the lemma. Suppose we have a representation

$$f = \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}$$

in $K[x_1, \dots, x_n]_{\mathbf{m}}$. By (2.3) and (2.4), we know there is a representation

$$f = \sum_{i=1}^n \tilde{h}_i \frac{\partial f}{\partial x_i}$$

with $\tilde{h}_1, \dots, \tilde{h}_n$ lying in the maximal ideal of $K[[x_1, \dots, x_n]]$. It follows that

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i} (h_i - \tilde{h}_i) = 0.$$

But since $\partial f / \partial x_1, \dots, \partial f / \partial x_n$ form a regular sequence in $K[x_1, \dots, x_n]_{\mathbf{m}}$, they also form a regular sequence in $K[[x_1, \dots, x_n]]$. Thus there exists a skew-symmetric set $\{\eta_{ij}\}_{i,j=1}^n \subset K[[x_1, \dots, x_n]]$ (i. e., $\eta_{ji} = -\eta_{ij}$) such that

$$h_i - \tilde{h}_i = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \eta_{ij}.$$

But this implies that h_1, \dots, h_n lie in the maximal ideal of $K[[x_1, \dots, x_n]]$, hence they must also lie in the maximal ideal of $K[x_1, \dots, x_n]_{\mathbf{m}}$.

3. Some reduction steps

We begin with some general remarks on the spectral sequence $E_t^{r,s}$. Let $\omega \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$ for some m , $0 \leq m \leq n-1$. If $\omega \in F_r \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$, we may write

$$\omega = \sum_{k=0}^r \omega^{(k)},$$

where $\omega^{(k)}$ is a homogeneous form of degree k . The assertion that $E_e^{r,m-r} = 0$ means that if

$$(3.1) \quad \sum_{j=0}^i df^{(\delta-j)} \wedge \omega^{(r-i+j)} = 0$$

for $i = 0, 1, \dots, e-1$, then there exist $\{\xi_j^{(r-\delta+j)}\}_{j=1}^e \subseteq \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{m-1}$, where $\xi_j^{(r-\delta+j)}$ is homogeneous of degree $r-\delta+j$, such that

$$(3.2) \quad \omega^{(r)} = \sum_{j=0}^{e-1} df^{(\delta-j)} \wedge \xi_{j+1}^{(r-\delta+j+1)}$$

and such that

$$(3.3) \quad \sum_{j=0}^i df^{(\delta-j)} \wedge \xi_{j+e-i}^{(r-\delta+j+e-i)} = 0$$

for $i = 0, 1, \dots, e-2$.

Now fix $e = \delta - \delta' + 1$. From (1.7), $f^{(\delta-j)} = 0$ for $0 < j < e-1$, thus (3.1) implies

$$(3.4) \quad df^{(\delta)} \wedge \omega^{(r)} = 0$$

and

$$(3.5) \quad df^{(\delta)} \wedge \omega^{(r-\delta+\delta')} + df^{(\delta')} \wedge \omega^{(r)} = 0$$

and (3.2) and (3.3) become

$$(3.6) \quad \omega^{(r)} = df^{(\delta)} \wedge \xi_1^{(r-\delta+1)} + df^{(\delta')} \wedge \xi_e^{(r-\delta'+1)}$$

and

$$(3.7) \quad df^{(\delta)} \wedge \xi_e^{(r-\delta'+1)} = 0.$$

The vanishing of $E_{\delta-\delta'+1}^{r,m-r}$ for all $m < n$ and all r is thus a consequence of the following stronger assertion.

PROPOSITION 3.8. *Assume the hypothesis of Theorem 1.10. If $\omega \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$, $m < n$, is a homogeneous form satisfying*

$$(3.9) \quad df^{(\delta)} \wedge \omega = 0$$

and

$$(3.10) \quad df^{(\delta')} \wedge \omega = df^{(\delta)} \wedge \xi$$

for some homogeneous form $\xi \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$, then there exists a homogeneous form $\eta \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{m-1}$ such that

$$(3.11) \quad \omega = df^{(\delta)} \wedge \eta.$$

Proof. The conclusion for $m < n - 1$ follows simply from the fact that $f^{(\delta)} = 0$ has only isolated singularities in \mathbf{P}^{n-1} . This implies that the ideal of $\mathbf{F}_q[x_1, \dots, x_n]$ generated by $\{\partial f^{(\delta)} / \partial x_i\}_{i=1}^n$ has height $n - 1$, therefore also has depth $n - 1$. It then follows directly from [11] that condition (3.9) alone implies the existence of the desired η satisfying (3.11). In other words, we have

$$(3.12) \quad H^m(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}, \phi_{f^{(\delta)}}) = 0 \quad \text{for } m < n - 1.$$

So assume that $\omega \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{n-1}$ is a homogeneous form satisfying (3.9) and (3.10) for some homogeneous form $\xi \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{n-1}$. We express (3.9) and (3.10) in coordinate form. Let

$$\begin{aligned} \omega &= \sum_{i=1}^n (-1)^{i-1} \omega_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \\ \xi &= \sum_{i=1}^n (-1)^{i-1} \xi_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \end{aligned}$$

where $\omega_i, \xi_i \in \mathbf{F}_q[x_1, \dots, x_n]$ are homogeneous polynomials. Then (3.9) becomes

$$(3.13) \quad \sum_{i=1}^n \frac{\partial f^{(\delta)}}{\partial x_i} \omega_i = 0$$

and (3.10) becomes

$$(3.14) \quad \sum_{i=1}^n \frac{\partial f^{(\delta')}}{\partial x_i} \omega_i = \sum_{i=1}^n \frac{\partial f^{(\delta)}}{\partial x_i} \xi_i.$$

To simplify the calculation, we make a coordinate change. Let $a_1, \dots, a_s \in \mathbf{P}^{n-1}$ be the singular points of $f^{(\delta)} = 0$. Since the generic hyperplane section of a hypersurface with isolated singularities is smooth, we can make a coordinate change on \mathbf{A}^n so that the hyperplane $x_n = 0$ in \mathbf{P}^{n-1} intersects the hypersurface $f^{(\delta)} = 0$ in \mathbf{P}^{n-1} transversally, in particular, the singularities a_1, \dots, a_s do not lie on $x_n = 0$. This implies that the polynomials $x_n, \partial f^{(\delta)} / \partial x_1, \dots, \partial f^{(\delta)} / \partial x_{n-1}$ taken in any order form a regular sequence in $\mathbf{F}_q[x_1, \dots, x_n]$. (We are using here the hypothesis that $(p, \delta) = 1$.)

We claim that it is enough to show that

$$(3.15) \quad \omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right)$$

in $\mathbf{F}_q[x_1, \dots, x_n]$. To see this, suppose

$$\omega_n = \sum_{i=1}^{n-1} h_i \frac{\partial f^{(\delta)}}{\partial x_i}$$

for some homogeneous polynomials h_i and substitute into (3.13) to get

$$\sum_{i=1}^{n-1} \frac{\partial f^{(\delta)}}{\partial x_i} \left(\omega_i + h_i \frac{\partial f^{(\delta)}}{\partial x_n} \right) = 0.$$

Since $\partial f^{(\delta)}/\partial x_1, \dots, \partial f^{(\delta)}/\partial x_{n-1}$ form a regular sequence, there exists a skew-symmetric set $\{\eta_{ij}\}_{i,j=1}^{n-1}$ of homogeneous polynomials such that

$$\omega_i + h_i \frac{\partial f^{(\delta)}}{\partial x_n} = \sum_{j=1}^{n-1} \eta_{ij} \frac{\partial f^{(\delta)}}{\partial x_j} \quad \text{for } i = 1, \dots, n-1.$$

If we set $\eta_{in} = -h_i$, $\eta_{ni} = h_i$, for $i = 1, \dots, n-1$ and $\eta_{nn} = 0$, then $\{\eta_{ij}\}_{i,j=1}^n$ is a skew-symmetric set satisfying

$$(3.16) \quad \omega_i = \sum_{j=1}^n \eta_{ij} \frac{\partial f^{(\delta)}}{\partial x_j} \quad \text{for } i = 1, \dots, n.$$

If we then define

$$\eta = \sum_{1 \leq i < j \leq n} (-1)^i \eta_{ij} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_n,$$

equation (3.16) implies equation (3.11).

The common zeros of $\partial f^{(\delta)}/\partial x_1, \dots, \partial f^{(\delta)}/\partial x_{n-1}$ in \mathbf{P}^{n-1} form a finite set containing the singular points of $f^{(\delta)} = 0$, so we may write this set as

$$\{a_1, \dots, a_s, b_1, \dots, b_t\}.$$

Since $x_n, \partial f^{(\delta)}/\partial x_1, \dots, \partial f^{(\delta)}/\partial x_{n-1}$ form a regular sequence, none of these points lies on the hypersurface $x_n = 0$. Note that our hypotheses imply that the hypersurface $\partial f^{(\delta)}/\partial x_n = 0$ in \mathbf{P}^{n-1} contains the points a_1, \dots, a_s but does not contain any of the points b_1, \dots, b_t . The main technical tool for proving (3.15) is the following.

LEMMA 3.17. *There exists a homogeneous polynomial $P \in \mathbf{F}_q[x_1, \dots, x_n]$ such that*

$$P\omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right)$$

and such that the hypersurface $P = 0$ in \mathbf{P}^{n-1} does not contain any of the points a_1, \dots, a_s .

Remark. The proof of Lemma 3.17 will require several steps. Before starting the proof, we explain how it implies (3.15). For any fixed $j \in \{1, \dots, t\}$, we can find a linear form $h_j \in \mathbf{F}_q[x_1, \dots, x_n]$ such that the hyperplane $h_j = 0$ in \mathbf{P}^{n-1} contains b_j but contains none of a_1, \dots, a_s . Multiplying P by such factors, we may assume in addition to the conclusion of the lemma that the hypersurface $P = 0$ in \mathbf{P}^{n-1} contains b_1, \dots, b_t . Choose nonnegative integers α, β such that $x_n^\alpha P + x_n^\beta \partial f^{(\delta)}/\partial x_n$ is homogeneous. The properties of P imply that the hypersurface $x_n^\alpha P + x_n^\beta \partial f^{(\delta)}/\partial x_n = 0$ in \mathbf{P}^{n-1} contains none of the points $a_1, \dots, a_s, b_1, \dots, b_t$. Thus the homogeneous polynomials

$$\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}}, x_n^\alpha P + x_n^\beta \frac{\partial f^{(\delta)}}{\partial x_n}$$

have no common zero in \mathbf{P}^{n-1} and hence form a regular sequence in $\mathbf{F}_q[x_1, \dots, x_n]$. But (3.13) and Lemma 3.17 imply that

$$\left(x_n^\alpha P + x_n^\beta \frac{\partial f^{(\delta)}}{\partial x_n} \right) \omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

This implies (3.15).

4. Proof of Lemma 3.17

There are two basic ideas involved in the proof of Lemma 3.17. The first is expressed in the following.

LEMMA 4.1. *For each singular point a_i , $i = 1, \dots, s$, there exist homogeneous polynomials $Q_i, R_1^{(i)}, \dots, R_{n-1}^{(i)}$ such that*

$$(4.2) \quad Q_i f^{(\delta)} = \sum_{j=1}^{n-1} R_j^{(i)} \frac{\partial f^{(\delta)}}{\partial x_j}$$

and such that a_i does not lie on the hypersurface $Q_i = 0$ in \mathbf{P}^{n-1} but does lie on all the hypersurfaces $R_j^{(i)} = 0$ for $j = 1, \dots, n-1$.

Proof. Fix i and let $(\tilde{a}_1, \dots, \tilde{a}_{n-1}, 1)$ be homogeneous coordinates for $a_i \in \mathbf{P}^{n-1}$. Put

$$\tilde{f}(y_1, \dots, y_{n-1}) = f^{(\delta)}(y_1, \dots, y_{n-1}, 1).$$

By Lemma 2.5 (we are using here the hypothesis that $(p, \delta_i) = 1$), we have

$$(4.3) \quad \tilde{f} = \sum_{j=1}^{n-1} \tilde{h}_j \frac{\partial \tilde{f}}{\partial y_j},$$

where $\tilde{h}_1, \dots, \tilde{h}_{n-1}$ lie in the maximal ideal of the local ring of $(\tilde{a}_1, \dots, \tilde{a}_{n-1})$, i. e., $\tilde{h}_j = \tilde{P}_j / \tilde{Q}_j$ where $\tilde{P}_j, \tilde{Q}_j \in K[y_1, \dots, y_{n-1}]$ and

$$\begin{aligned} \tilde{Q}_j(\tilde{a}_1, \dots, \tilde{a}_{n-1}) &\neq 0 \\ \tilde{P}_j(\tilde{a}_1, \dots, \tilde{a}_{n-1}) &= 0 \end{aligned}$$

for $j = 1, \dots, n-1$. Multiplying (4.3) by $\tilde{Q} := \tilde{Q}_1 \cdots \tilde{Q}_{n-1}$ gives a relation

$$(4.4) \quad \tilde{Q} \tilde{f} = \sum_{j=1}^{n-1} \tilde{R}_j \frac{\partial \tilde{f}}{\partial y_j}$$

in $\mathbf{F}_q[y_1, \dots, y_{n-1}]$ with $\tilde{Q}(\tilde{a}_1, \dots, \tilde{a}_{n-1}) \neq 0$ and $\tilde{R}_j(\tilde{a}_1, \dots, \tilde{a}_{n-1}) = 0$ for $j = 1, \dots, n-1$. Making the substitution $y_j \mapsto x_j/x_n$ in (4.4) and multiplying by a sufficiently high power of x_n then gives the desired assertion.

By the argument used in the remark following Lemma 3.17, we may assume in addition to the conclusion of Lemma 4.1 that the hypersurfaces $Q_i = 0$, $R_1^{(i)} = 0$, \dots , $R_{n-1}^{(i)} = 0$ in \mathbf{P}^{n-1} all contain the points $a_1, \dots, \hat{a}_i, \dots, a_s$. Choose nonnegative integers $\alpha_1, \dots, \alpha_s$ such that

$$Q = x_n^{\alpha_1} Q_1 + \dots + x_n^{\alpha_s} Q_s$$

is homogeneous. Multiplying (4.2) by $x_n^{\alpha_i}$ and summing over i then gives the following.

COROLLARY 4.5. *There exist homogeneous polynomials Q, R_1, \dots, R_{n-1} such that*

$$Q f^{(\delta)} = \sum_{j=1}^{n-1} R_j \frac{\partial f^{(\delta)}}{\partial x_j}$$

and such that the hypersurface $Q = 0$ in \mathbf{P}^{n-1} contains none of the points a_1, \dots, a_s and the hypersurfaces $R_j = 0$ contain the points a_1, \dots, a_s for $j = 1, \dots, n-1$.

Multiplying the Euler relation for $f^{(\delta)}$ by Q gives

$$\delta Q f^{(\delta)} = \sum_{j=1}^n x_j Q \frac{\partial f^{(\delta)}}{\partial x_j}.$$

Combined with Corollary 4.5, this gives

$$(4.6) \quad x_n Q \frac{\partial f^{(\delta)}}{\partial x_n} = \sum_{j=1}^{n-1} S_j \frac{\partial f^{(\delta)}}{\partial x_j},$$

where

$$(4.7) \quad S_j = \delta R_j - x_j Q \quad \text{for } j = 1, \dots, n-1.$$

We can now prove Lemma 3.17. Multiplying (3.13) by $x_n Q$ and using (4.6) leads to

$$\sum_{j=1}^{n-1} \frac{\partial f^{(\delta)}}{\partial x_j} (x_n Q \omega_j + S_j \omega_n) = 0.$$

Since $\partial f^{(\delta)}/\partial x_1, \dots, \partial f^{(\delta)}/\partial x_{n-1}$ form a regular sequence, there exists a skew-symmetric set $\{\eta_{ij}\}_{i,j=1}^{n-1}$ of homogeneous polynomials such that

$$(4.8) \quad x_n Q \omega_j + S_j \omega_n = \sum_{k=1}^{n-1} \eta_{jk} \frac{\partial f^{(\delta)}}{\partial x_k} \quad \text{for } j = 1, \dots, n-1.$$

Multiplying (3.14) by $x_n Q$ and using (4.6) gives

$$\sum_{j=1}^n \frac{\partial f^{(\delta')}}{\partial x_j} x_n Q \omega_j \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

Substitution from (4.8) then gives

$$(4.9) \quad \left(- \sum_{j=1}^{n-1} \frac{\partial f^{(\delta')}}{\partial x_j} S_j + \frac{\partial f^{(\delta')}}{\partial x_n} x_n Q \right) \omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

We now come to the second basic idea of the proof. Put

$$P = - \sum_{j=1}^{n-1} \frac{\partial f^{(\delta')}}{\partial x_j} S_j + \frac{\partial f^{(\delta')}}{\partial x_n} x_n Q,$$

a homogeneous polynomial. By (4.9), P satisfies the first assertion of Lemma 3.17. We show that it satisfies the second assertion as well. Let (c_1, \dots, c_n) be a set of homogeneous coordinates for one of the points a_1, \dots, a_s . By (4.7) and Corollary 4.5, we see that

$$\begin{aligned} P(c_1, \dots, c_n) &= \sum_{j=1}^n c_j Q(c_1, \dots, c_n) \frac{\partial f^{(\delta')}}{\partial x_j}(c_1, \dots, c_n) \\ &= \delta' Q(c_1, \dots, c_n) f^{(\delta')}(c_1, \dots, c_n) \end{aligned}$$

using the Euler relation for $f^{(\delta')}$. By hypothesis $\delta' f^{(\delta')}(c_1, \dots, c_n) \neq 0$ and by Corollary 4.5 $Q(c_1, \dots, c_n) \neq 0$, hence $P(c_1, \dots, c_n) \neq 0$. This proves Lemma 3.17, which completes the proof of Theorem 1.10.

5. Proof of Theorem 1.11

Throughout this section, we assume the hypothesis of Theorem 1.11. Since the generic hyperplane section of a hypersurface with isolated singularities is smooth, we may assume, after a coordinate change if necessary, that the hyperplane $x_n = 0$ intersects $f^{(\delta)} = 0$ transversally. Let $\tilde{f} \in \mathbf{F}_q[x_1, \dots, x_{n-1}]$ be defined by

$$\tilde{f}(x_1, \dots, x_{n-1}) = f^{(\delta)}(x_1, \dots, x_{n-1}, 0).$$

Then $\tilde{f} = 0$ defines a smooth hypersurface in \mathbf{P}^{n-2} .

LEMMA 5.1. *Under the above conditions, $\partial f^{(\delta)}/\partial x_1, \dots, \partial f^{(\delta)}/\partial x_{n-1}$ form a regular sequence.*

Proof. It suffices to show that $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^{n-1}$ have only finitely many common zeros in \mathbf{P}^{n-1} . Since $p|\delta$, the Euler relation becomes

$$(5.2) \quad x_n \frac{\partial f^{(\delta)}}{\partial x_n} = - \sum_{i=1}^{n-1} x_i \frac{\partial f^{(\delta)}}{\partial x_i},$$

thus any common zero of $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^{n-1}$ is a zero of either $\partial f^{(\delta)}/\partial x_n$ or x_n . Those which are zeros of $\partial f^{(\delta)}/\partial x_n$ form a finite set by the hypothesis of Theorem 1.11. Those which are zeros of x_n are in one-to-one correspondence with the zeros of $\{\partial \tilde{f}/\partial x_i\}_{i=1}^{n-1}$ in \mathbf{P}^{n-2} . Since $\tilde{f} = 0$ defines a smooth hypersurface in \mathbf{P}^{n-2} , this set must also be finite.

To prove Theorem 1.11, the discussion in section 3 shows that it suffices to prove the analogue of Proposition 3.8.

PROPOSITION 5.3. *Assume the hypothesis of Theorem 1.11. If $\omega \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$, $m < n$, is a homogeneous form satisfying*

$$(5.4) \quad df^{(\delta)} \wedge \omega = 0$$

and

$$(5.5) \quad df^{(\delta')} \wedge \omega = df^{(\delta)} \wedge \xi$$

for some homogeneous form $\xi \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$, then there exists a homogeneous form $\eta \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{m-1}$ such that

$$(5.6) \quad \omega = df^{(\delta)} \wedge \eta.$$

Proof. By Lemma 5.1, the ideal of $\mathbf{F}_q[x_1, \dots, x_n]$ generated by $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n$ has depth $n-1$. Thus, just as in the proof of Proposition 3.8, we conclude that for $m < n-1$, (5.4) alone implies (5.6). Hence (3.12) holds under the hypothesis of Theorem 1.11 also.

Suppose now $m = n-1$. As in the proof of Proposition 3.8, we write (5.4) and (5.5) in coordinate form:

$$(5.7) \quad \sum_{i=1}^n \frac{\partial f^{(\delta)}}{\partial x_i} \omega_i = 0$$

$$(5.8) \quad \sum_{i=1}^n \frac{\partial f^{(\delta')}}{\partial x_i} \omega_i = \sum_{i=1}^n \frac{\partial f^{(\delta)}}{\partial x_i} \xi_i.$$

The same argument as before (see (3.15)) reduces us to proving that

$$(5.9) \quad \omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

LEMMA 5.10. *Suppose that (5.7) holds and that*

$$f^{(\delta')} \omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

Then

$$\omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

Proof. The zeros of $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n$ in \mathbf{P}^{n-1} form a finite set $\{a_1, \dots, a_s\}$ and the zeros of $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^{n-1}$ form a finite set $\{a_1, \dots, a_s, b_1, \dots, b_t\}$. For $i = 1, \dots, t$, choose a linear form $h_i \in \mathbf{F}_q[x_1, \dots, x_n]$ that vanishes at b_i but not at a_j for any j . Let $k \in \mathbf{F}_q[x_1, \dots, x_n]$ be a linear form that does not vanish at any b_i . For suitably chosen nonnegative integers α and β , the polynomial

$$g := (h_1 \cdots h_t)^\alpha f^{(\delta')} + k^\beta \frac{\partial f^{(\delta)}}{\partial x_n} \in \mathbf{F}_q[x_1, \dots, x_n]$$

is homogeneous. By the hypothesis of Theorem 1.11, $f^{(\delta')}$ does not vanish at a_i for any i . It follows that g does not vanish at any a_i or b_j , i. e., the polynomials

$$\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}}, g$$

have no common zero in \mathbf{P}^{n-1} , hence they form a regular sequence. But the hypothesis of the lemma implies that

$$g\omega_n \in (\partial f^{(\delta)}/\partial x_1, \dots, \partial f^{(\delta)}/\partial x_{n-1}).$$

The conclusion of the lemma now follows from the defining property of regular sequences.

By Lemma 5.10, we are reduced to showing the following.

LEMMA 5.11. *If (5.7) and (5.8) hold, then*

$$f^{(\delta')} \omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

Proof. Multiplying (5.7) by x_n and substituting from the Euler relation (5.2) gives

$$\sum_{i=1}^{n-1} \frac{\partial f^{(\delta)}}{\partial x_i} (x_n \omega_i - x_i \omega_n) = 0.$$

By Lemma 5.1, this implies

$$(5.12) \quad x_n \omega_i - x_i \omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right)$$

for $i = 1, \dots, n$. Multiplying (5.8) by x_n and using (5.2) gives

$$(5.13) \quad \sum_{i=1}^n x_n \omega_i \frac{\partial f^{(\delta')}}{\partial x_i} \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

It follows from (5.12) and (5.13) that

$$\sum_{i=1}^n x_i \omega_n \frac{\partial f^{(\delta')}}{\partial x_i} \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

The Euler relation for $f^{(\delta')}$ now implies

$$\delta' f^{(\delta')} \omega_n \in \left(\frac{\partial f^{(\delta)}}{\partial x_1}, \dots, \frac{\partial f^{(\delta)}}{\partial x_{n-1}} \right).$$

The conclusion of the lemma then follows from the hypothesis that $(p, \delta') = 1$.

6. Formula for M_f

By [4, section 1], we know that if $E_{\delta-\delta'+1}^{r,s} = 0$ for all r, s with $r + s \neq n$, then

$$(6.1) \quad M_f = \dim_{\mathbf{F}_q} \left(\bigoplus_{r+s=n} E_{\delta-\delta'+1}^{r,s} \right).$$

We describe the terms on the right-hand side explicitly.

For $0 \leq m \leq n$, let $H^m(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}})^{(r)}$ denote the homogeneous component of degree r of $H^m(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}})$ relative to the grading on $\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$ defined in section 1. We define a map

$$(6.2) \quad \phi_{f^{(\delta')}} : H^m(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}}) \rightarrow H^{m+1}(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}}).$$

Let $\omega \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$ be such that $df^{(\delta)} \wedge \omega = 0$ and let $[\omega] \in H^m(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}})$ be the cohomology class of ω . We define

$$(6.3) \quad \phi_{f^{(\delta')}}([\omega]) = [df^{(\delta')} \wedge \omega].$$

From the definition of the spectral sequence $E_t^{r,s}$, one sees that $E_{\delta-\delta'+1}^{r,n-r}$ is the cokernel of the map

$$\phi_{f^{(\delta')}} : H^{n-1}(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}})^{(r-\delta'+\delta)} \rightarrow H^n(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}})^{(r)}.$$

It follows from (6.1) that

$$(6.4) \quad M_f = \dim_{\mathbf{F}_q} (\text{coker}(\phi_{f^{(\delta')}} : H^{n-1}(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}}) \rightarrow H^n(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}}))).$$

We compute the dimension of this cokernel.

Under the hypothesis of either Theorem 1.10 or 1.11, $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n$ have finitely many common zeroes in \mathbf{P}^{n-1} , say, a_1, \dots, a_s . By a coordinate change, we may assume a_1, \dots, a_s lie in the open set $x_n \neq 0$, which we identify with \mathbf{A}^{n-1} . Put

$$h = f^{(\delta)}(y_1, \dots, y_{n-1}, 1) \in \mathbf{F}_q[y_1, \dots, y_{n-1}].$$

The Milnor number μ_i of a_i is given by

$$\mu_i = \dim_{\mathbf{F}_q} \mathbf{F}_q[y_1, \dots, y_{n-1}]_{\mathbf{m}_i} / (\partial h / \partial y_1, \dots, \partial h / \partial y_{n-1}),$$

where $\mathbf{F}_q[y_1, \dots, y_{n-1}]_{\mathbf{m}_i}$ denotes the localization of $\mathbf{F}_q[y_1, \dots, y_{n-1}]$ at the maximal ideal \mathbf{m}_i corresponding to a_i .

PROPOSITION 6.5. *Under the hypothesis of either Theorem 1.10 or 1.11,*

$$M_f = (\delta - 1)^n - (\delta - \delta') \sum_{i=1}^s \mu_i.$$

Proof. The graded module $H^i(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{\bullet}, \phi_{f^{(\delta)}})$ has a Poincaré series $p_i(t)$:

$$p_i(t) = \sum_{r=0}^{\infty} \left(\dim_{\mathbf{F}_q} H^i(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{\bullet}, \phi_{f^{(\delta)}})^{(r)} \right) t^r.$$

Using only $\deg f = \delta$, one has always

$$\sum_{i=0}^n (-1)^{n-i} p_i(t) = \frac{(1 - t^{\delta-1})^n}{(1 - t)^n}.$$

Under the hypothesis of either Theorem 1.10 or 1.11, equation (3.12) holds. Thus $p_i(t) = 0$ for $i < n - 1$ and we have

$$(6.6) \quad p_n(t) - p_{n-1}(t) = \frac{(1 - t^{\delta-1})^n}{(1 - t)^n}.$$

Put $f_n = \partial f^{(\delta)} / \partial x_n$ and define

$$h_n = f_n(y_1, \dots, y_{n-1}, 1) \in \mathbf{F}_q[y_1, \dots, y_{n-1}].$$

The Euler relation for $f^{(\delta)}$ implies

$$(6.7) \quad \delta h = h_n + \sum_{i=1}^{n-1} y_i \frac{\partial h}{\partial y_i}.$$

The proof of Choudary-Dimca[5, Corollary 9] shows that for all sufficiently large r , $\dim_{\mathbf{F}_q} H^n(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{\bullet}, \phi_{f^{(\delta)}})^{(r)}$ is a constant equal to

$$(6.8) \quad \sum_{i=1}^s \dim_{\mathbf{F}_q} \mathbf{F}_q[y_1, \dots, y_{n-1}]_{\mathbf{m}_i} / (\partial h / \partial y_1, \dots, \partial h / \partial y_{n-1}, h_n).$$

Since the right-hand side of (6.6) is a polynomial in t , it follows that for sufficiently large r , $\dim_{\mathbf{F}_q} H^{n-1}(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{\bullet}, \phi_{f^{(\delta)}})^{(r)}$ also equals (6.8). If $p \nmid \delta$, (6.7) shows that h_n lies in the ideal generated by $\{\partial h / \partial y_i\}_{i=1}^{n-1}$. If a_i is a weighted homogeneous isolated singular point, then h lies in the ideal of $\mathbf{F}_q[y_1, \dots, y_{n-1}]_{\mathbf{m}_i}$ generated by $\{\partial h / \partial y_i\}_{i=1}^{n-1}$, so by (6.7) h_n also lies in this ideal. Thus in either case, (6.8) equals $\sum_{i=1}^s \mu_i$. We can summarize these facts by saying that there exist polynomials $q_n(t), q_{n-1}(t)$ such that

$$(6.9) \quad p_n(t) = \frac{q_n(t)}{1 - t}, \quad p_{n-1}(t) = \frac{q_{n-1}(t)}{1 - t},$$

and such that

$$(6.10) \quad q_n(1) = q_{n-1}(1) = \sum_{i=1}^s \mu_i.$$

By Propositions 3.8 and 5.3 the mapping (6.2) is injective for $m \leq n - 1$, and it is homogeneous of degree $\delta' - \delta$ in the grading we have defined, hence the Poincaré series of its cokernel is

$$\begin{aligned} p_n(t) - t^{\delta' - \delta} p_{n-1}(t) &= p_n(t) - p_{n-1}(t) + (1 - t^{\delta' - \delta}) p_{n-1}(t) \\ &= p_n(t) - p_{n-1}(t) + \frac{t^{\delta - \delta'} - 1}{1 - t} \frac{q_{n-1}(t)}{t^{\delta - \delta'}} \end{aligned}$$

using (6.9). By (6.6), this expression simplifies to

$$(1 + t + \cdots + t^{\delta-2})^n - (1 + t + \cdots + t^{\delta-\delta'-1}) \frac{q_{n-1}(t)}{t^{\delta-\delta'}}.$$

Using (6.10), we see that the value of this polynomial at $t = 1$ is

$$(\delta - 1)^n - (\delta - \delta') \sum_{i=1}^s \mu_i,$$

which completes the proof of Proposition 6.5.

References

- [1] A. Adolphson and S. Sperber, *Exponential sums and Newton polyhedra: Cohomology and estimates*, Ann. of Math. **130** (1989), 367–406.
- [2] ———, *On the zeta function of a complete intersection*, Ann. Sci. E. N. S. **29** (1996), 287–328.
- [3] ———, *Exponential sums on \mathbf{A}^n* , Israel J. Math. (to appear)
- [4] ———, *Exponential sums on \mathbf{A}^n , II* (preprint)
- [5] A. Choudary and A. Dimca, *Koszul complexes and hypersurface singularities*, Proc. A. M. S. **121** (1994), 1009–1016.
- [6] P. Deligne, *La conjecture de Weil, I*, Publ. Math. I. H. E. S. **43** (1974), 273–307.
- [7] P. Deligne and N. Katz, *Groupes de Monodromie en Géométrie Algébrique (SGA 7 II)*, Lecture Notes in Math. v. 340, Springer-Verlag, Berlin-Heidelberg-New York, 1973.
- [8] B. Dwork, *On the zeta function of a hypersurface*, Publ. Math. I. H. E. S. **12** (1962), 5–68.
- [9] R. García López, *Exponential sums and singular hypersurfaces*, Manuscripta Math. **97** (1998), 45–58.
- [10] E. Looijenga, *Isolated Singular Points on Complete Intersections*, London Math. Soc. Lecture Note Series v. 77, Cambridge University Press, Cambridge, 1984.
- [11] K. Saito, *On a generalization of de Rham lemma*, Ann. Inst. Fourier, Grenoble **26** (1976), 165–170.

DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74078

E-mail address: `adolphs@math.okstate.edu`

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455

E-mail address: `sperber@math.umn.edu`